BIAXIAL UNSTEADY STRAIN OF A RECTILINEAR STRIP IN THE SCHEME OF A COMPRESSIBLE VISCOPLASTIC MEDIUM

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Solutions [1-3] describing the tension of a viscoplastic rectilinear strip of incompressible material with a linear velocity field are known. The solutions of a mathematical model are constructed in the papers mentioned without taking account of the inertial terms in the motion equations of the incompressible medium. Now, if the mathematical calculations are duplicated for a known linear velocity field, then by taking the inertial terms into account we arrive at the deduction that no nonzero solution exists. The unsteady tension of a rectilinear strip in the scheme of an incompressible medium with internal strength, zero tangential stress, and linear velocity field is not the solution of the motion equations with inertial terms taken into account. Let us note that in the case of tension of a strip in the scheme of an incompressible ideal fluid, such a solution exists [4].

In this paper the exact solution of a mathematical model for unsteady strain of a rectilinear strip under tension is determined with a linear velocity field and with zero tangential stress in the scheme of a compressible viscoplastic medium. Analytical dependences are deduced to estimate the strip rupture time. The existence of a plasticity peak is noted by analogy with [5, 6].

1. Mathematical Model

The stress tensor components σ_{11}, σ_{12}, σ_{22}, the velocity vector components v_1, v_2 along the rectangular coordinate axes in the x_1, x_2 plane, the density of ρ of a continuous compressible medium are determined from the solution of the following relationships for an arbitrary closed domain:

Equations of motion of a medium outside the field of external mass forces

$$\partial\sigma_{ij}/\partial x_j = \rho(\partial v_i/\partial t + v_j\partial v_i/\partial x_j); \tag{1.1}$$

Continuity equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0, \tag{1.2}$$

where the repeated subscripts i, j assume summation, and $t \geqslant 0$ is the time.

According to the hypotheses formulated in [1], the relationships of a viscoplastic compressible body up to the assumption of incompressibility of the medium in the plane case have the form

$$\begin{gathered}
\sigma_{11} = \sigma + \mu\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \frac{\sigma_s}{2}\cos 2\varphi,\ \sigma = \frac{1}{2}(\sigma_{11} + \sigma_{22}),\\
\sigma_{22} = \sigma - \mu\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) - \frac{\sigma_s}{2}\cos 2\varphi,\\
\sigma_{12} = \mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) + \frac{\sigma_s}{2}\sin 2\varphi,\\
\operatorname{tg} 2\varphi = \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\Big/\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right),
\end{gathered} \tag{1.3}$$

where φ is the angle between the direction corresponding to the greatest principal stress and the x_1 axis, σ_S is the dynamic yield point, and μ is the coefficient of dynamic viscosity. In the case of an incompressible medium, the system (1.3) agrees with the known relationships [2] for a rigidly plastic body ($\mu = 0$) and a viscous fluid ($\sigma_S = 0$).

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Assuming the unsteady motion being considered for the compressible medium to be isentropic, the closing equation of state is representable in the form of a given barotropic function. The most customary form of writing the shock compressibility law for metals which sets up the connection between the mean pressure p and the density ρ has the form (see [7], for instance) is

$$p = A[(\rho/\rho_0)^n - 1], \ \sigma = -p, \tag{1.4}$$

where A, n > 0 are constants to be determined by test.

Let $F(t, x_1, x_2) = 0$ be the equation of the boundary of the domain under consideration. We require satisfaction of the kinematic

$$\partial F/\partial t + v_i \partial F/\partial x_i = 0 \tag{1.5}$$

and the dynamic condition

$$P_j = \sigma_{ij} n_i \ (i, j = 1, 2), \tag{1.6}$$

on this boundary, where P_j is the projection of stresses acting on the boundary of the deformable domain, and n_i are the direction cosines of the outer normal to the boundary with respect to the coordinate axes.

A domain of a continuous compressible medium with its boundary F_0, the initial velocity field and the density ρ_0 are given at the initial time t = 0.

Together with the initial data and the equation of state, the system (1.1)-(1.6) defines a closed mathematical model for the unsteady motion of a compressible viscoplastic medium in a domain with variable boundary.

2. Tension of a Rectilinear Strip

We henceforth assume that the domain under consideration is a rectilinear strip which conserves its boundaries as lines in the whole time interval considered for the motion. In this case the equation of the boundary takes the form $F(t, x_1, x_2) = \{(x_i \pm a_i(t) = 0\}$, where for definiteness $2a_1$ is the length, and $2a_2$ is the height of the strip. Let us consider tension of a strip with zero tangential stress ($\sigma_{12} = 0$). We also formulate the following assumption: during unsteady strain of a compressible viscoplastic medium the density is homogeneous in the domain under consideration, and is a function of just the time $\rho = \rho(t)$.

Because $\sigma_{12} = 0$, from (1.3) there follows that

$$\partial v_1/\partial x_2 + \partial v_2/\partial x_1 = 0. \tag{2.1}$$

Let us consider a sufficiently smooth stream function $\psi(t, x_1, x_2)$ which is defined by the formulas

$$v_1 = \partial\psi/\partial x_1, \ v_2 = -\partial\psi/\partial x_2, \tag{2.2}$$

hence, (2.1) is satisfied identically.

Taking (2.2) into account as well as the fact that the density of the medium is a function of just the time, we obtain from (1.2) and (1.3), respectively

$$\frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2} = -\frac{1}{\rho}\frac{d\rho}{dt}; \tag{2.3}$$

$$\sigma_{11} = \sigma + \mu\Delta\psi + \sigma_s/2, \ \Delta\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2}, \tag{2.4}$$

$$\sigma_{22} = \sigma - \mu\Delta\psi - \sigma_s/2, \ \sigma = \sigma(t), \ \sigma_{12} = 0.$$

Substituting (2.4) into (1.1), we obtain two nonlinear differential equations in the stream function because of (2.2). Differentiating the first equation with respect to the variable x_2, and the second with respect to x_1, and adding the results, we arrive at the equation $\partial^2(\Delta\psi)/(\partial x_1 \partial x_2) = 0$, which can be written in the form

$$\Delta\psi = C_1(t, x_1) + C_2(t, x_2), \tag{2.5}$$

where $C_1(t, x_1)$ and $C_2(5, x_2)$ are unknown functions of their arguments.

From the kinematic condition (1.5), (2.2), we have the following relationships on the boundary

$$\partial\psi/\partial x_1 = \dot{a}_1 \quad \text{for} \quad x_1 = a_1, \quad -\partial\psi/\partial x_2 = \dot{a}_2 \quad \text{for} \quad x_2 = a_2. \tag{2.6}$$

Here and below the dot above a quantity corresponds to differentiation with respect to time.

As follows from (1.) and (2.2)-(2.6), one of the nontrivial solutions for the viscoplastic strip is tension of the latter with a linear velocity field. In this case we have for the stream function

$$\psi(t, x_1, x_2) = \frac{1}{2}\left[c_1(t)\, x_1^2 + c_2(t)\, x_2^2\right], \tag{2.7}$$

hence we obtain because of (2.3) and (2.6)

$$c_2 - c_1 = \dot{\rho}/\rho, \quad a_i = a_{i0} \exp\left(\pm \int\limits_0^t c_i(\tau)\, d\tau\right). \tag{2.8}$$

After substituting (2.4) and (2.7) into (1.1), we obtain for the functions $c_i(5)$ of the time

$$\dot{c}_1 + c_1^2 = 0, \; \dot{c}_2 - c_2^2 = 0,$$

whose solutions have the form

$$c_1 = c_{10}(1 + c_{10}t)^{-1}, \quad c_2 = c_{20}(1 - c_{20}t)^{-1}. \tag{2.9}$$

According to (2.8) and (2.9), the functional dependences in time with respect to the density and the law of variation of the strip boundary are expressed by the formulas

$$\rho = \rho_0/(1 + c_{10}t)(1 - c_{20}t); \tag{2.10}$$

$$a_1 = a_{10}(1 + c_{10}t), \quad a_2 = a_{20}(1 - c_{20}t), \tag{2.11}$$

where $c_{10} \geqslant 0$, $c_{20} \geqslant 0$ are constants with dimensionality 1/c determining the gradient of the strain rate of the boundary.

We henceforth introduce dimensionless variables and parameters by means of the relationships

$$\sigma_i = \sigma_{ii}/\rho_0 V_{10}^2, \; \bar{\sigma}_s = \sigma_s/\rho_0 V_{10}^2, \; \bar{A} = A/\rho_0 V_{10}^2,$$
$$\nu = \mu/\rho_0 V_{10} a_{10}, \; \theta = \rho/\rho_0, \; \bar{t} = c_{20}t, \; \gamma = c_{10}/c_{20}, \; c_{20} > 0, \tag{2.12}$$

where $V_{10} = c_{10}a_{10}$ is the rate of strip tension along the horizontal at the initial time. The bar above the dimensionless quantities is omitted below. Then because of (2.9) and (2.12), we have from (1.4), (2.4) and (2.10)

$$\theta = [(1 + \gamma t)(1 - t)]^{-1}; \tag{2.13}$$

$$\sigma_1 = -A(\theta^n - 1) + \nu\theta(1 + \gamma^{-1}) + \sigma_s/2,$$
$$\sigma_2 = -A(\theta^n - 1) - \nu\theta(1 + \gamma^{-1}) - \sigma_s/2. \tag{2.14}$$

At the time t = 1 the height of the strip is zero, but the density becomes infinite. The formulas (2.13)-(2.14) have physical meaning in the time segment $t \in [0, 1]$. We hence note that the functional dependence (2.13) has an extremum: for $\gamma > 1$ from (2.13) we obtain $\theta_{min} = 4/(2 + \gamma + \gamma^{-1})$ for $t = (\gamma - 1)/2\gamma$. In case $\gamma \in [0, 1]$, the density of the strip increases monotonically in the whole time segment $t \in [0, 1]$. A numerical computation of the change in strip material density with time by means of (2.13) is represented in Fig. 1 for different values of $\gamma > 0$. Curves 1-6 in Fig. 1 correspond to γ = 10; 5; 2; 1; 0.4; 0.2. In particular, it follows from (2.14) that for $\gamma > 1$ the stress components σ_i in the strip and on the boundary have extremal values in the time segment $t \in [0, 1]$ when the initial parameters A, ν, σ_s are fixed.

Under the effect of explosive loads real metals increase their density. Equation (1.4) is obtained for metals under the assumption $\theta > 1$ for $t > 0$. For the above to be satisfied for the rectilinear strip under consideration, the range of variation $\gamma \in (0, 1]$ must be required for the initial parameter γ. In this case, the density and stress increase monotonically during tension of the strip, as follows from (2.13) and (2.14). A numerical computation using (2.14) is represented in Figs. 2 and 3 for the change in stress σ_i on the strip with time for the fixed parameters A = 2.74, n = 5.5, ν = 0.05, σ_s = 0.04. The numbers 1-4 on the graphs correspond to γ = 1; 0.8; 0.4; 0.2. According to (2.12), the dimensionless

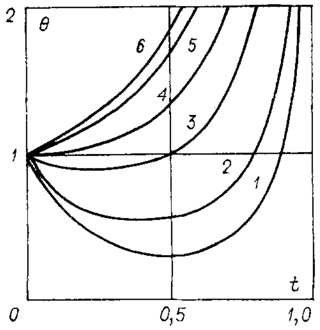

Fig. 1

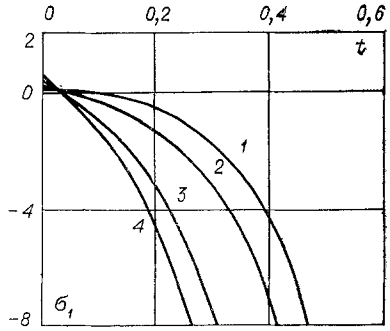

Fig. 2

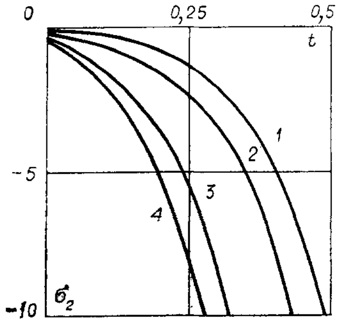

Fig. 3

parameters are obtained for a steel strip which is stretched for the following values of the dimensional quantities: σ_s = 0.34 GPa, $\rho_0 = 7.85 \cdot 10^3$ kg/m^3, a_{10} = 0.1 m, a_{20} = 0.01 m, V_{10} = 1000 m/sec, $\mu = 4 \cdot 10^4$ kg/(m·sec), A = 21.5 GPa, n = 5.5. The last two parameters from (1.4) are obtained for steel, according to [7], in the 0-100 GPa pressure range. The dynamic viscosity coefficient for low-carbon steels are found from [8].

The graphs in Figs. 2 and 3 show the homogeneity of the compressive stress field in the strip for t > 0, whose intensity increases with time. To estimate the boundary loads P_i (i = 1, 2), we have the relationship $\sigma_1 = -P_1$, $\sigma_2 = -P_2$ from (1.6).

Therefore, an exact solution is obtained for the problem of unsteady motion of a rectilinear strip in a compressible viscoplastic medium scheme. A strip with a linear velocity field is stretched at a constant rate by homogeneous loads on the boundaries. The solution is obtained under the assumption of no tangential stress in the strip. The density of the medium is here a function of just the time.

3. Rupture of the Strip

Let us examine the solution obtained above for a strip in the case of its rupture under tension. For total rupture of a solid along the section under consideration, the following should be satisfied [9]: the time criterion for preparation of the body to rupture and the integrated time criterion of total closing of cracks on the basis of the nonstationary crack growth equation. The strip tension energy being liberated here is expended in crack development. A dynamic criterion for crack closing and total rupture reduces to an integral relation [6], which takes the following form for a rectilinear strip

$$\int_0^{t_*} cqa_1 dt = \frac{\alpha a_{2*}}{1-k} \ln(1/S_0),\ a_{2*} = a_{20}(1 - c_{20}t_*), \tag{3.1}$$

where t_* is the rupture time, q is the energy density being liberated by a sound wave, c is the speed of sound, α is the work of formation of unit crack area, k is the average coefficient of reflection of the acoustic wave energy flux with respect to time, and S_0 is the initial fraction of the area of the body section under consideration which overlaps the cracks. The effective energy of dynamic rupture $\alpha_* = (\alpha \ln S_0)/(k-1)$ is determined from experiment.

According to [10], the estimate of energy density for plane strain is realized by means of the formula

$$q = T^2(1 - \nu_0^2)/2E, \tag{3.2}$$

where T is the stress intensity, ν_0 is the Poisson ratio, and E is Young's modulus. We turn to dimensionless quantities according to (2.12). We use the notation

$$T_1 = T/\rho_0 V_{10}^2, \ \tau_* = c_{20} t_*, \tag{3.3}$$

then we obtain from (3.1)-(3.3)

$$\int_0^{\tau_*} T_1^2(t)(1+\gamma t)\,dt = 2\beta(1-\tau_*)/\gamma, \ \beta = \frac{\alpha_* E a_{20}}{(1-\nu_0^2)\, c a_{10}^2 \rho_0^2 V_{10}^3} \tag{3.4}$$

In the case of plane strain of a strip where the stresses σ_1, σ_2, $\sigma_3 = (1/2)(\sigma_1 + \sigma_2)$ occur and there are no tangential stresses, we obtain a formula from (2.14) for the stress intensity invariant

$$T_1 = \sigma_s/2 + \nu(1+\gamma^{-1})\theta. \tag{3.5}$$

After substituting (3.5) into (3.4) and integrating, we obtain

$$\gamma\sigma_s^2\tau_*(2+\gamma\tau_*)/8 - \nu\sigma_s(1 + \gamma)\ln(1-\tau_*) + \nu^2\left[\frac{\tau_*(1+\gamma^{-1})}{1-\tau_*} + \ln\frac{(1+\gamma\tau_*)}{(1-\tau_*)}\right] = 2\beta(1-\tau_*). \tag{3.6}$$

To first order accuracy in $\tau_* \ll 1$, from (3.6) we have the following simple relationship for the time of strip rupture

$$\tau_* = 2\beta/(2\beta + \gamma[\sigma_s/2 + \nu(1+\gamma^{-1})]^2). \tag{3.7}$$

Analyzing (3.7) with respect to the parameter $\gamma \in (0, 1]$, we note that τ_* has the maximum

$$\tau_{max} = \beta/[\beta + \nu(\sigma_s + 2\nu)] \ \text{for} \ \gamma = \nu/(\sigma_s/2 + \nu). \tag{3.8}$$

This feature of the dynamic behavior of viscoplastic media was first obtained and interpreted by test data in the case of explosive deformation of tubes in [5]. The existence of a dynamic plasticity peak during tension on a viscoplastic rod in a zero-dimensional formulation is shown in [6]. Therefore, the relationships (3.7) and (3.8) verify the deduction formulated in [5] that the dynamic plasticity peak is general in nature in the rupture of shells and the simplest structures fabricated from viscoplastic materials.

As an illustration, we compute the rupture time of a steel strip with the following dimensional parameters: $\sigma_S = 0.34$ GPa; $\mu = 4\cdot10^4$ kg/(m·sec); $\rho_0 = 7.85\cdot10^3$ kg/m^3; $V_{10} = 1000$ m/sec; $a_{10} = 0.10$; $a_{20} = 0.01$; E = 200 GPa; $\alpha_* = 2\cdot10^5$ kg/sec^2; A = 21.5 GPa; n = 5.5; $\nu_0 = 0.3$. The speed of sound in the strip is estimated by the formula $c^2 = dp/d\rho$ for $\rho = \rho_0$ in conformity with (1.4). Hence, $c^2 = An/{}_0$ or c = 3873 m/sec. Going over to dimensionless quantities according to (2.12) and (3.3), we obtain $\tau_{max} = 0.72$ for $\gamma = 0.71$, $\beta = 0.18\cdot10^{-3}$ from (3.8) and (3.9). Taking into account that $c_{10}/c_{20} = 0.71$, $t_{max} = \tau_{max}/c_{20}$, where $c_{10} = 10^4$ 1/sec, from (3.9) we have the maximum strip rupture time in dimensional form $t_{max} = 51$ µsec.

4. Dissipative Function

Let us introduce the rate of mechanical energy dissipation per unit volume W. It is known [11] that under real strain of an arbitrary domain in a continuous medium, the dissipative function is positive everywhere in the domain. Let us consider W for the solution obtained above. From (2.2) and (2.4) we have

$$W = \sigma_{11}\frac{\partial^2\psi}{\partial x_1^2} - \sigma_{22}\frac{\partial^2\psi}{\partial x_2^2}.$$

Taking into account (2.7) and (2.12)-(2.14), we obtain in dimensionless form

$$W = A(\theta^n - 1)\dot{\theta}/\theta + (1+\gamma^{-1})\theta[\nu(1+\gamma^{-1})\theta + \sigma_s/2]. \tag{4.1}$$

From an analysis of this latter expression for n > 0 there follows that $W \geqslant 0$ everywhere in the strip when $t \in [0, 1]$, $\gamma \in (0,1]$. In case $\gamma > 1$ this assertion is not evident. Let us examine the first component in (4.1). Taking into account the equalities

$$\theta^n - 1 = (\theta - 1)(\theta^{n-1} + \theta^{n-2} + \ldots + 1),\ \dot{\theta}/\theta = c_2 - c_1,$$

we arrive at the following estimate for the sign: $(\theta^n - 1)\dot{\theta}/\theta \geqslant 0$ for $t \in [0, (1/2)(1 - \gamma^{-1})]$ and $t \in [(1 - \gamma^{-1}), 1]$. In the intermediate interval $t \in ((1/2)(1 - \gamma^{-1}), 1 - \gamma^{-1})$ the expression under consideration is negative. In this case the general estimate of the sign of W depends on the relationship between the two members in (4.1). It is important to note that the dissipative function is positive for any $\gamma > 0$ in the initial state of the strain of a viscoplastic strip.

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INVESTIGATION OF MATERIAL DAMAGE UNDER CREEP AND CREEP STRENGTH

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Within the framework of the mechanics of continuous media, the conception of the mechanical equation of state with a system of kinetic equations to determine the parameters q_i characterizing the state under consideration

$$\dot{p} = \dot{p}(\sigma, T, q_1, q_2, \ldots, q_n); \tag{1}$$

$$dq_j = a_j d\sigma + b_j dT + c_j dt,\ j = 1, 2, \ldots, n. \tag{2}$$

which has been proposed in [1], is often the starting point to describe metal creep. According to (1), the creep rate is determined by the stress σ, the temperature T, and a certain number of structural parameters q_j. In the general case, (2) represent nonintegrable kinetic relationships to describe changes in the parameters q_j, which in turn characterize a change in the material structure (a_j, b_j, c_j are certain functions of σ, T, t, as well as of q_j). In order to describe at least certain qualitative features of the creep strength of metals, one structural parameter ω is most often introduced for simplicity, and it is taken as a certain measure of material damage. In solving creep and creep strength problems, usually either the physical meaning of the parameter ω is not made specific, or ω is understood to be the relative part of the specimen section damaged as a result of creep. The rupture time $t = t^*$ is often understood to be the time at which the damage reaches unity ($\omega^* = \omega(t^*) = 1$). In

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